

g^* -Angle between two subspaces in the space of p -summable sequences

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g^* -Angle Between Two Subspaces in The Space of p -Summable Sequences

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Abstract. We define a new semi-inner product g^* on l^p spaces for $1 \leq p < \infty$ equipped with a 2-norm. Using g^* , we study the g^* -angle between two vectors on l^p . We also develop the notion of the g^* -angle between a 1-dimensional subspace and an k -dimensional subspace for $k \geq 1$ in the 2-normed space.

INTRODUCTION

Let $(V, \|\cdot\|)$ be a real normed space. The functional $g: V \times V \rightarrow \mathbb{R}$ defined by

$$g(u, v) := \frac{1}{2} \|u\| [\tau_+(u, v) + \tau_-(u, v)]$$

where

$$\tau_{\pm}(u, v) := \lim_{h \rightarrow 0^{\pm}} \frac{\|u + hv\| - \|u\|}{h}.$$

We know that $g(u, v)$ satisfies the following basic properties [5]:

- 1) $g(u, u) = \|u\|^2$ for any $u \in V$;
- 2) $g(\alpha u, \beta v) = \alpha\beta g(u, v)$ for any $u, v \in V$ and $\alpha, \beta \in \mathbb{R}$;
- 3) $g(u, u + v) = \|u\|^2 + g(u, v)$ for any $u, v \in V$;
- 4) $|g(u, v)| \leq \|u\| \|v\|$ for any $u, v \in V$.

In general, $g(u, v)$ is not linear in v . If, in addition to the above three properties $g(u, v)$ is linear in v , then g is called a *semi-inner product* on V . For instance, the function

$$g(u, v) = \|u\|_p^{2-p} \sum_j |u_j|^{p-1} \operatorname{sgn}(u_j) v_j, \quad u := (u_j), v := (v_j) \in l^p \quad (1)$$

is a semi-inner product on $(l^p, \|\cdot\|_p)$ for $1 \leq p < \infty$ [5].

¹

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Many researchers have studied g -angle between two vectors in V using the semi-inner product g , see, for instance [1,9,10,11]. In 2018, Nur *et al.* [14] develop the notion of the g -angle between two subspaces of V . If $S = \text{span}\{s\}$ is a 1-dimensional subspace and $T = \text{span}\{t_1, \dots, t_k\}$ is a k -dimensional subspace of V with $k \geq 1$, then the g -angle between subspaces S and T is defined by $A_g(S, T)$ with $\cos^2 A_g(S, T) = \frac{(g(s_T, s))^2}{\|s\|^2 \|s_T\|^2}$. In the formula, s_T is the g -orthogonal projection of s on T . Moreover, Nur *et al.* show that the value of $\cos A_g(S, T)$ is equal to the ratio between the 'length' of the g -orthogonal projection of s on T and the 'length' of s ($\cos^2 A_g(S, T) = \frac{\|s_T\|^2}{\|s\|^2}$). Recently, Nur and Gunawan [13] define the angle between 2-dimensional subspaces by using a 2-norm.

In general, a 2-norm on a vector space V is a mapping $\|\cdot, \cdot\|: V \times V \rightarrow \mathbb{R}$ which satisfies the following four conditions:

- (1) $\|u, v\| = 0$ if and only if u, v are linearly dependent;
- (2) $\|u, v\| = \|v, u\|$ for any $u, v \in V$;
- (3) $\|\alpha u, v\| = |\alpha| \|u, v\|$, for any $u, v \in V$ and for any $\alpha \in \mathbb{R}$;
- (4) $\|u, v + w\| \leq \|u, v\| + \|u, w\|$, for any $u, v, w \in V$.

Next, the part $(V, \|\cdot, \cdot\|)$ is called a 2-normed space.

Geometrically, 2-norm $\|u, v\|$ may be interpreted as the area of the 2-dimensional parallelepiped spanned by u, v . The notion of 2-normed space was first developed in the mid 1960's by Gähler [4]. Recent results can be found, for instance, in [2,3,7,8,12]. On the space l^p for $1 \leq p < \infty$, the following the 2-norm was defined by Gunawan in [6],

$$\|u, v\|_p = \left[\frac{1}{2} \sum_j \sum_k (\text{abs} \begin{vmatrix} u_j & u_k \\ v_j & v_k \end{vmatrix})^p \right]^{\frac{1}{p}} \quad (2)$$

where $u := (u_j), v := (v_j) \in l^p$.

In this article, we will define a semi-inner product g^* on $(l^p, \|\cdot, \cdot\|_p)$ with $1 \leq p < \infty$. Next, using a semi-inner product g^* , we can introduce the g^* -angle between two vectors. Moreover, we can study g^* -angle between a 1-dimensional subspace and a k -dimensional subspace for $k \geq 1$ in the 2-normed space $(l^p, \|\cdot, \cdot\|_p)$.

MAIN RESULTS

g^* -Angle Between Two Vectors

In this subsection, we will discuss g^* -angle between two vectors in the 2-normed space $(l^p, \|\cdot, \cdot\|_p)$. Let $\{a_1, a_2\}$ be a linearly independent set on l^p . Firstly, we define the following function.

$$\|u\|_p^* = [\|u, a_1\|_p^p + \|u, a_2\|_p^p]^{\frac{1}{p}}, \quad (3)$$

for every $u \in l^p$. Next, we have the following proposition.

Proposition 2.1. [6] *The mapping $\|\cdot\|_p^*$ defines a norm on l^p .*

Using the norm $\|\cdot\|_p^*$ with $a_1 = (a_{1j})$ and $a_2 = (a_{2j})$, we define a mapping $g^*(\cdot, \cdot)$ on the 2-normed space $(l^p, \|\cdot, \cdot\|_p)$ with $1 \leq p < \infty$ by

$$g^*(u, v) = \frac{(\|u\|_p^*)^{2-p}}{2} \sum_{i=1}^2 \sum_j \sum_k (\text{abs} \begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik} \end{vmatrix})^{p-1} (\text{sgn} \begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik} \end{vmatrix}) \begin{vmatrix} v_j & v_k \\ a_{ij} & a_{ik} \end{vmatrix} \quad (4)$$

for every $u = (u_j), v = (v_j) \in l^p$.

Then we have the following result.

Theorem 2.2. *The mapping $g^*(u, v)$ in (4) defines a semi-inner product on $(l^p, \|\cdot\|_p)$.*

Proof. We will show that $g^*(u, v)$ satisfies the properties of g and linear in terms of v .

1. Observe that

$$\begin{aligned} g^*(u, u) &= \frac{(\|u\|_p^*)^{2-p}}{2} \sum_{i=1}^2 \sum_j \sum_k (\text{abs} \begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik} \end{vmatrix})^{p-1} (\text{sgn} \begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik} \end{vmatrix}) \begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik} \end{vmatrix} \\ &= (\|u\|_p^*)^{2-p} [\|u, a_1\|_p^p + \|u, a_2\|_p^p] \\ &= (\|u\|_p^*)^2. \end{aligned}$$

2. Observe that

$$\begin{aligned} g^*(\alpha u, \beta v) &= \frac{(\|\alpha u\|_p^*)^{2-p}}{2} \sum_{i=1}^2 \sum_j \sum_k (\text{abs} \begin{vmatrix} \alpha u_j & \alpha u_k \\ a_{ij} & a_{ik} \end{vmatrix})^{p-1} (\text{sgn} \begin{vmatrix} \alpha u_j & \alpha u_k \\ a_{ij} & a_{ik} \end{vmatrix}) \begin{vmatrix} \beta v_j & \beta v_k \\ a_{ij} & a_{ik} \end{vmatrix} \\ &= \alpha \beta \frac{(\|u\|_p^*)^{2-p}}{2} \sum_{i=1}^2 \sum_j \sum_k (\text{abs} \begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik} \end{vmatrix})^{p-1} (\text{sgn} \begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik} \end{vmatrix}) \begin{vmatrix} v_j & v_k \\ a_{ij} & a_{ik} \end{vmatrix} \\ &= \alpha \beta g^*(u, v). \end{aligned}$$

3. Using properties of determinants, we obtain

$$\begin{aligned} g^*(u, u + v) &= \frac{(\|u\|_p^*)^{2-p}}{2} \sum_{i=1}^2 \sum_j \sum_k (\text{abs} \begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik} \end{vmatrix})^{p-1} (\text{sgn} \begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik} \end{vmatrix}) \begin{vmatrix} u_j + v_j & u_k + v_k \\ a_{ij} & a_{ik} \end{vmatrix} \\ &= \frac{(\|u\|_p^*)^{2-p}}{2} \sum_{i=1}^2 \sum_j \sum_k (\text{abs} \begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik} \end{vmatrix})^{p-1} (\text{sgn} \begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik} \end{vmatrix}) \begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik} \end{vmatrix} \\ &\quad + \frac{(\|u\|_p^*)^{2-p}}{2} \sum_{i=1}^2 \sum_j \sum_k (\text{abs} \begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik} \end{vmatrix})^{p-1} (\text{sgn} \begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik} \end{vmatrix}) \begin{vmatrix} v_j & v_k \\ a_{ij} & a_{ik} \end{vmatrix} \\ &= (\|u\|_p^*)^2 + g^*(u, v). \end{aligned}$$

4. Observe that

$$\begin{aligned} |g^*(u, v)| &= \left| \frac{(\|u\|_p^*)^{2-p}}{2} \sum_{i=1}^2 \sum_j \sum_k (\text{abs} \begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik} \end{vmatrix})^{p-1} (\text{sgn} \begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik} \end{vmatrix}) \begin{vmatrix} v_j & v_k \\ a_{ij} & a_{ik} \end{vmatrix} \right| \\ &\leq \frac{(\|u\|_p^*)^{2-p}}{2} \sum_{i=1}^2 \sum_j \sum_k (\text{abs} \begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik} \end{vmatrix})^{p-1} (\text{abs} \begin{vmatrix} v_j & v_k \\ a_{ij} & a_{ik} \end{vmatrix}) \\ &\leq (\|u\|_p^*)^{2-p} \left[\frac{1}{2} \sum_{i=1}^2 \sum_j \sum_k (\text{abs} \begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik} \end{vmatrix})^p \right]^{\frac{p-1}{p}} \left[\frac{1}{2} \sum_{i=1}^2 \sum_j \sum_k (\text{abs} \begin{vmatrix} v_j & v_k \\ a_{ij} & a_{ik} \end{vmatrix})^p \right]^{\frac{1}{p}} \\ &= \|u\|_p^* \|v\|_p^*. \end{aligned}$$

5. Observe that

$$g^*(u, v + v') = \frac{(\|u\|_p^*)^{2-p}}{2} \sum_{i=1}^2 \sum_j \sum_k (\text{abs} \begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik} \end{vmatrix})^{p-1} (\text{sgn} \begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik} \end{vmatrix}) \begin{vmatrix} v_j + v'_j & v_k + v'_k \\ a_{ij} & a_{ik} \end{vmatrix}$$

$$\begin{aligned}
&= \frac{(\|u\|_p^*)^{2-p}}{2} \sum_{i=1}^2 \sum_j \sum_k (\text{abs} \begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik} \end{vmatrix})^{p-1} (\text{sgn} \begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik} \end{vmatrix}) \begin{vmatrix} v_j & v_k \\ a_{ij} & a_{ik} \end{vmatrix} \\
&\quad + \frac{(\|u\|_p^*)^{2-p}}{2} \sum_{i=1}^2 \sum_j \sum_k (\text{abs} \begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik} \end{vmatrix})^{p-1} (\text{sgn} \begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik} \end{vmatrix}) \begin{vmatrix} v_j' & v_k' \\ a_{ij} & a_{ik} \end{vmatrix} \\
&= g^*(u, v) + g^*(u, v').
\end{aligned}$$

Therefore, $g^*(u, v)$ defines the semi-inner product on $(l^p, \|\cdot\|_p)$. ■

Remark 2.3. The functional $g^*(\cdot; \cdot)$ does not satisfy commutative property. For example, consider l^1 with $a_1 = (1, 0, 0, \dots)$, $a_2 = (0, 1, 0, \dots)$ and $\|\cdot\|_1^*$ in Proposition 2.1. Take $u = (1, 1, 3, 0, \dots)$ and $v = (2, 1, -3, 0, \dots)$. Clearly $\|u\|_1^* = 8$ and $\|v\|_1^* = 9$. Thus $g^*(v, u) = -45 \neq g^*(u, v) = -16$.

Next, using (4) and definition of g -angle in [14], we have the g^* -angle between vectors u and v on $(l^p, \|\cdot\|_p)$ as follows.

$$A_g^*(u, v) = \arccos \frac{g^*(v, u)}{\|u\|_p^* \|v\|_p^*}. \quad (5)$$

Note that $A_g^*(u, v) = \frac{1}{2}\pi$ if and only if $g^*(v, u) = 0$ or $v \perp_g u$.

Example 2.4. Let $(l^1, \|\cdot\|_1)$ be 2 normed space with $a_1 = (1, 0, 0, \dots)$, and $a_2 = (0, 1, 0, \dots)$. If $u = (1, 2, 1, 0, \dots)$ and $v = (2, 1, 3, 0, \dots)$ then $\|u, a_1\|_1 = 3$, $\|u, a_2\|_1 = 2$, $\|y, a_1\|_1 = 4$ and $\|v, a_2\|_1 = 5$. Clearly $\|u\|_1^* = 5$ and $\|v\|_1^* = 9$. Moreover, using g^* in Theorem 2.2 for $p = 1$, we obtain

$$\begin{aligned}
g^*(v, u) &= \frac{\|v\|_p^*}{2} \sum_{i=1}^2 \sum_j \sum_k (\text{sgn} \begin{vmatrix} v_j & v_k \\ a_{ij} & a_{ik} \end{vmatrix}) \begin{vmatrix} u_j & u_k \\ a_{ij} & a_{ik} \end{vmatrix} \\
&= \frac{5}{2}(10) = 25.
\end{aligned}$$

Hence, $A_g^*(u, v) = \arccos \left(\frac{5}{9}\right)$.

g^* -Angle Between a 1-Dimensional Subspace and an k -Dimensional Subspace

In this part, we will define the g^* -angle between a 1-dimensional subspace and an k -dimensional subspace in $(l^p, \|\cdot\|_p)$. We will apply the definition of g -angle in [14] with a new norm $\|\cdot\|_p^*$ that is obtained from the 2-norm. Using $g^*(\cdot; \cdot)$ in (4), we have the Gram determinant $\Gamma_{g^*}(t_1, \dots, t_n) = \mathbf{det}[g^*(t_i, t_j)]$, where $g^*(t_i, t_j)$ is the j -th element of the i -th row. Moreover, we have the g^* -orthogonal projection of s on T as follows.

Definition 2.5 Let s be a vector of l^p and $T = \text{span}(t_1, \dots, t_n)$ be subspace of l^p with $\Gamma_{g^*}(t_1, \dots, t_n) \neq 0$. The g^* -orthogonal projection of s on T (s_{T^*}) is defined by

$$s_{T^*} = -\frac{1}{\Gamma_{g^*}(t_1, \dots, t_n)} \begin{vmatrix} 0 & t_1 & \dots & t_n \\ g^*(t_1, s) & g^*(t_1, t_1) & \dots & g^*(t_1, t_n) \\ \vdots & \vdots & \ddots & \vdots \\ g^*(t_n, s) & g^*(t_n, t_1) & \dots & g^*(t_n, t_n) \end{vmatrix}.$$

By using s_{T^*} in Definition 2.8, we have g^* -orthogonal complement $s_{T^*}^\perp$ as follows.

$$s_{T^\perp}^\perp = s - s_{T^*} = \frac{1}{\Gamma_{g^*}(t_1, \dots, t_n)} \begin{vmatrix} s & t_1 & \dots & t_n \\ g^*(t_1, s) & g^*(t_1, t_1) & \dots & g^*(t_1, t_n) \\ \vdots & \vdots & \ddots & \vdots \\ g^*(t_n, s) & g^*(t_n, t_1) & \dots & g^*(t_n, t_n) \end{vmatrix}.$$

Next, we write the g^* -angle between $S = \text{span}(s)$ and $T = \text{span}(t_1, \dots, t_k)$ for $k \geq 1$ of $(l^p, \|\cdot\|_p)$ as follows.

Definition 2.6. Let $(l^p, \|\cdot\|_p)$ be a 2-normed space for $1 \leq p < \infty$. If $S = \text{span}(s)$ is a 1-dimensional subspace and $T = \text{span}(t_1, \dots, t_k)$ is an k -dimensional subspace of l^p with $\Gamma_{g^*}(t_1, \dots, t_k) \neq 0$ dan $k \geq 1$ then g^* -angle between S and T denoted by $A_g^*(S, T)$ with

$$\cos^2 A_g^*(S, T) = \frac{(g^*(s_{T^*}, s))^2}{(\|s_{T^*}\|_p^*)^2 (\|s\|_p^*)^2}, \quad (6)$$

where s_{T^*} is the g^* -orthogonal projection of s on T .

Next, by using the formula $s = s_{T^*} + s_{T^\perp}^\perp$, we have the value of $\cos^2 A_g^*(S, T)$ by

$$\begin{aligned} \cos^2 A_g^*(S, T) &= \frac{(g^*(s_{T^*}, s_{T^*} + s_{T^\perp}^\perp))^2}{(\|s_{T^*}\|_p^*)^2 (\|s\|_p^*)^2} = \frac{(g^*(s_{T^*}, s_{T^*}) + g^*(s_{T^*}, s_{T^\perp}^\perp))^2}{(\|s_{T^*}\|_p^*)^2 (\|s\|_p^*)^2} \\ &= \frac{(g^*(s_{T^*}, s_{T^*}))^2}{(\|s_{T^*}\|_p^*)^2 (\|s\|_p^*)^2} = \frac{(\|s_{T^*}\|_p^*)^2}{(\|s\|_p^*)^2}. \end{aligned}$$

Example 2.7. Let $(l^1, \|\cdot\|_1)$ be a 2-normed space with $a_1 = (1, 0, 0, \dots)$, and $a_2 = (0, 1, 0, \dots)$. If $S = \text{span}(s)$ and $T = \text{span}(t_1, t_2)$ with $s = (1, 1, 3, 0, \dots)$, $t_1 = (1, 0, 0, 0, \dots)$ and $t_2 = (0, 1, 0, 0, \dots)$ then $\|s\|_1^* = 8$, $\|t_1\|_1^* = 1$, and $\|t_2\|_1^* = 1$. Using the semi inner product- g in Theorem 2.2, we have

$$\Gamma_{g^*}(t_1, t_2) = \begin{vmatrix} g^*(t_1, t_1) & g^*(t_1, t_2) \\ g^*(t_2, t_1) & g^*(t_2, t_2) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

As a consequence, we obtain

$$\begin{aligned} s_{T^*} &= -\frac{1}{\Gamma_{g^*}(t_1, t_2)} \begin{vmatrix} 0 & t_1 & t_2 \\ g^*(t_1, s) & g^*(t_1, t_1) & g^*(t_1, t_2) \\ g^*(t_2, s) & g^*(t_2, t_1) & g^*(t_2, t_2) \end{vmatrix} \\ &= -\begin{vmatrix} 0 & t_1 & t_2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = t_1 + t_2 = (1, 1, 0, \dots). \end{aligned}$$

Next, we can compute $\|s_{T^*}\|_1^*$ by

$$\begin{aligned} \|s_{T^*}\|_1^* &= \frac{1}{2} \sum_j \sum_k \left(\text{abs} \begin{vmatrix} s_{T^*j} & s_{T^*k} \\ a_{1j} & a_{1k} \end{vmatrix} \right) + \frac{1}{2} \sum_j \sum_k \left(\text{abs} \begin{vmatrix} s_{T^*j} & s_{T^*k} \\ a_{2j} & a_{2k} \end{vmatrix} \right) \\ &= \frac{1}{2} \sum_j \left(\text{abs} \begin{vmatrix} s_{T^*j} & 1 \\ a_{1j} & 1 \end{vmatrix} \right) + \left(\text{abs} \begin{vmatrix} s_{T^*j} & 1 \\ a_{1j} & 0 \end{vmatrix} \right) + \frac{1}{2} \sum_j \left(\text{abs} \begin{vmatrix} s_{T^*j} & 1 \\ a_{1j} & 0 \end{vmatrix} \right) + \left(\text{abs} \begin{vmatrix} s_{T^*j} & 1 \\ a_{2j} & 1 \end{vmatrix} \right) \\ &= 1 + 1 = 2. \end{aligned}$$

Thus $\cos^2 A_g^*(S, T) = \frac{4}{64}$, so that $A_g^*(S, T) = \arccos\left(\frac{1}{4}\right)$.

CONCLUDING REMARKS

We can extend our result for L^p spaces. Let V be a measure space and f is measurable function. The function $f \in L^p(V)$ if $\int_V |f(v)|^p dv < \infty$. Next, the function $\|f\|_{L^p} = \left(\int_V |f(v)|^p dv\right)^{\frac{1}{p}}$ defines a norm on $L^p(V)$. Gunawan [6] defines a 2-norm $\|\cdot, \cdot\|_{L^p}$ on $L^p(X) \times L^p(X)$ by

$$\|f_1, f_2\|_{L^p} = \left[\frac{1}{2} \int_V \int_V \text{abs} \left(\begin{vmatrix} f_1(v_1) & f_1(v_2) \\ f_2(v_1) & f_2(v_2) \end{vmatrix} \right) dv_1 dv_2 \right]^{\frac{1}{p}} \quad (7)$$

According to the 2-norm in (7), Ekariani et.al [3] defines a new norm on $L^p(V)$ by

$$\|f\|_{L^p}^* = [\|f, a_1\|_{L^p}^p + \|f, a_2\|_{L^p}^p]^{\frac{1}{p}},$$

where $\{a_1, a_2\}$ be a linearly independent set on $L^p(V)$. Using the norm $\|\cdot\|_{L^p}^*$, one may define $g^*(\cdot, \cdot)$ on $(L^p(V), \|\cdot\|_{L^p}^*)$ by

$$g^*(f_1, f_2) = \frac{(\|f_1\|_{L^p}^*)^{2-p}}{2} \sum_{i=1}^2 \int_X \int_X \left(\text{abs} \begin{vmatrix} f_1(v_1) & f_1(v_2) \\ a_i(v_1) & a_i(v_2) \end{vmatrix} \right)^{p-1} \left(\text{sgn} \begin{vmatrix} f_1(v_1) & f_1(v_2) \\ a_i(v_1) & a_i(v_2) \end{vmatrix} \right) \begin{vmatrix} f_2(v_1) & f_2(v_2) \\ a_i(v_1) & a_i(v_2) \end{vmatrix} dv_1 dv_2.$$

and check that this mapping defines a semi inner product on $L^p(V)$. The analogues in (5), we have the g^* -angle two vectors on $L^p(V)$. Moreover, the analogues in (6), we have the g^* -angle in a 2-normed space $(L^p(V), \|\cdot\|_{L^p}^*)$.

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